

# Indeterminacy with Constant Money Growth Rules and Income-Based Liquidity Constraints

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## Abstract

We study the implications of constant money growth rules on the stability properties of the equilibrium, in economies where agents are subject to a partial cash-in-advance constraint applying simultaneously to consumption and investment purchases. By reference to similar models in which the liquidity constraint applies only to consumption, we show that the inclusion of investment has dramatic, but contrasting, effects on the range of values giving rise to indeterminacy. First, it increases strongly a *lower bound* on the share of purchases requiring cash, below which the steady state is always indeterminate. Second, it creates a *higher bound* on this share, above which the steady-state is always determinate. In this context, the steady state value of the velocity of money becomes a crucial parameter for gauging whether constant money growth rules may be stabilizing or destabilizing for the economy.

*Keywords:* Cash-in-advance, indeterminacy, business cycles.

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# 1 Introduction

The stabilizing role of constant money growth rules in dynamic cash-in-advance economies has been the subject of strong debate and controversy in the literature, since Woodford (1994) demonstrated the possibility of endogenous fluctuations under such rules within a simple economy with cash and credit goods.<sup>1</sup> As the economy studied by Woodford (1994) is very simple, several extensions have been considered in ongoing work to assess the robustness of these indeterminacy results under liquidity constraints. In particular, using a partial cash-in-advance framework in the spirit of Grandmont and Younès (1972), Bosi and Magris (2003) study the emergence of sunspot fluctuations in a dynamic rational expectations economy with capital accumulation and fixed labor supply. In this paper, indeterminacy is shown to prevail for any value of the intertemporal elasticity of substitution, provided that the share of consumption purchases requiring cash-in-advance is below a certain threshold. When the share is above that threshold, indeterminacy arises only when the relative risk aversion of households is high enough (greater than 2, for example, in the standard Cooley and Hansen (1989) model with full liquidity constraint on consumption). As these values typically fall within the (admittedly imprecise) range of empirically plausible estimates for that parameter, these findings tend to suggest that constant money growth rules may be ineffective in ruling out indeterminacy and sunspot driven fluctuations when financial imperfections such as liquidity constraints are at stake in the economy.

However, in an interesting independent work, Carlstrom and Fuerst (2003) provide a model which challenges strongly this conclusion. Using a functional equivalence property between liquidity constraints and money-in-the utility function illustrated by Feenstra (1986), they provide a dynamic monetary model in which the rigid cash-in-advance constraint applying on consumption can be viewed as a limit case of perfect complementarity between real balances and consumption in the utility function. The advantage of their specification is that money demand depends, in this case, not only on consumption, but also on the nominal interest rate, a feature which is desirable from an empirical perspective. In that framework, it is shown that indeterminacy quickly disappears as soon as the interest-elasticity of money demand increases. As their economy is locally determinate for realistic estimates of this elasticity, Carlstrom and Fuerst (2003) conclude that money growth targeting rules are likely to have stabilizing properties on the economy.

This paper does not aim to solve the issue of the sensitivity of indeterminacy results in cash-in-advance economies when allowing for a non-zero interest rate elasticity of money demand. But it recognizes that the functional equivalence which is central to the results in Carlstrom and Fuerst (2003) is restrictive to the case in which the cash-in-advance constraint applies only to consumption purchases. From an empirical purpose, assuming that liquidity constraints con-

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<sup>1</sup>See Lucas and Stokey (1987) for a more detailed presentation of this model. For a similar approach of indeterminacy issues within models with money-in-the-utility-function, see Matsuyama (1990).

cern only consumption purchases may be seen as implausible as assuming a zero interest-elasticity of money demand. But as soon as liquid assets are also needed to finance a share of investment streams (as the business cycle literature on investment strongly emphasizes), the functional equivalence illustrated by Feenstra (1986) does not apply, and results derived within a money-in-the-utility-function framework are not extendable to economies in which agents are submitted to liquidity constraints.

For these reasons, this paper wishes to investigate the robustness of the indeterminacy results under constant money growth rules, when the liquidity constraint restricting agents decisions applies to a fraction of both consumption and investment purchases. We study whether this inclusion of investment in the liquidity constraint modifies strongly the range of parameters values giving rise to indeterminacy, by comparison to the reference economy studied in Bosi and Magris (2003) in which this constraint applies only to consumption. Our results show that *it does*, but in a way which is contrasted for the indeterminacy region: The inclusion of investment increases strongly a *lower bound* on the share of good purchases requiring cash below which the steady state is always indeterminate, but it creates a *higher bound* on this share above which the steady-state is always determinate. The stabilizing or destabilizing effects of constant money growth rule on the economy therefore depend crucially on the strength of the liquidity constraints – a parameter which, in the model, can also be interpreted as the (inverse of) the steady-state velocity of money. Empirical estimates for that parameter suggest that the economy have been in the determinacy zone during the recent decades, but that the observed upward trend of velocity in the US economy may quickly lead to a reversal of this conclusion.

The remainder of the paper is organized as follows. Section 2 sets up the model economy, derives the intertemporal equilibrium and characterizes the (unique) steady-state. Section 3 provides the stability analysis, and discusses the conditions for local indeterminacy. Section 4 concludes.

## 2 The model

We consider a discrete-time one-sector economy populated by a continuum of identical long-lived agents acting under perfect foresight, the size of which is normalized to one, and a representative firm producing under constant returns to scale.

**Consumers.** The representative consumer maximizes the discounted stream of utility functions  $\sum_{t=0}^{\infty} \beta^t u(c_t)$ , where  $\beta \in (0, 1)$  is the discount factor,  $c$  the consumption demand and the per-period utility function  $u$  satisfies the following basic restrictions:

**Assumption 1** The single-period utility function  $u(c)$  is twice continuously differentiable for all positive values of  $c$  and satisfies, for any  $c > 0$ ,  $u'(c) > 0$ ,  $u''(c) < 0$ ,  $\lim_{c \rightarrow 0} u'(c) = +\infty$  and  $\lim_{c \rightarrow +\infty} u'(c) = 0$ .

In each period  $t$ , households face a dynamic budget constraint  $B_t \equiv r_t k_t + w_t + (M_t + \tau_t)/p_t - (c_t + k_{t+1} - \Delta k_t + M_{t+1}/p_t) \geq 0$ , where  $p$  denotes the price of the good,  $k$  the physical equipment,  $M$  the money balances,  $r$  the real rental price of capital,  $w$  the real wage,  $\Delta \equiv 1 - \delta$  with  $\delta \in [0, 1]$  the depreciation rate of capital, and  $\tau$  the nominal lump-sum transfers issued by the government. For simplicity, labor supply is assumed to be inelastic:  $l_t = 1$  for every  $t \geq 0$ .<sup>2</sup> We suppose in addition that an amount  $q \in (0, 1]$  of consumption and investment purchases must be paid by cash in the hands of the representative consumer. In other words, agents are subject to an additional cash-in-advance constraint

$$C_t \equiv M_t/p_t - q(c_t + k_{t+1} - \Delta k_t) \geq 0 \quad (1)$$

The consumer maximizes the Lagrangian  $\Lambda_0 \equiv \sum_{t=0}^{\infty} \beta^t (u(c_t) + \lambda_t B_t + \nu_t C_t)$ , with respect to  $\{M_{t+1}, k_{t+1}, c_t\}_{t=0}^{\infty}$ , where  $\lambda$  and  $\nu$  are non-negative multipliers associated to, respectively, the budget constraint and the cash-in-advance. The FOC's for the representative household write:

$$\lambda_t = \beta(\lambda_{t+1} + \nu_{t+1})p_t/p_{t+1} \quad (2)$$

$$u'(c_t) = \lambda_t + q\nu_t \quad (3)$$

$$\lambda_t + q\nu_t = \beta[(\Delta + r_{t+1})\lambda_{t+1} + q\Delta\nu_{t+1}] \quad (4)$$

According to the arbitrage condition (2), the price of money at time  $t$ ,  $\lambda_t$ , is equal to its expected value in the following period plus the expected value of the implicit dividends  $\nu_{t+1}$  it will pay off. At the same time,  $\lambda_t$  can be viewed as the marginal indirect utility of real income in period  $t$ . However, as (3) establishes, at the optimum it does not equalize the marginal utility of consumption, since the individual cannot transform income into consumption, unless part of the former was previously held in form of money balances. Condition (4) is the intertemporal consumption-saving decision, which embodies the fractional liquidity constraint on physical investment.

In addition, the transversality condition must be satisfied:

$$\lim_{t \rightarrow +\infty} \beta^t u'(c_t) (k_{t+1} + \pi_{t+1} m_{t+1}) = 0 \quad (5)$$

where  $m_t \equiv M_t/p_t$  are the real balances held by the representative agent at the outset of period  $t - 1$  and  $\pi_{t+1} \equiv p_{t+1}/p_t$  is the gross inflation factor between period  $t$  and period  $t + 1$ .

**Firms.** The (aggregate) representative firm produces the good by mean of a standard constant returns to scale technology:  $Y = F(K, L)$ , where  $K$  and  $L$  stand, respectively, for aggregate capital and labor.

**Assumption 2** *The production function  $F : R_+^2 \rightarrow R_+$  is twice continuously differentiable, increasing in each argument, concave, homogeneous of de-*

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<sup>2</sup>We have also considered the case of an endogenous labor supply decision in an appendix available from the authors upon request. Analytical results are much more complicated combinations of the structural parameters in this case.

gree one and such that for any  $y > 0$ ,  $\lim_{x \rightarrow 0} F_1(x, y) = \lim_{x \rightarrow 0} F_2(y, x) = +\infty$ ,  
 $\lim_{x \rightarrow +\infty} F(x, y) = \lim_{x \rightarrow +\infty} F_2(y, x) = 0$ .

Profit maximization implies that in each period  $t$  the real interest rate and the real wage equalize, respectively, the marginal productivity of aggregate capital and the marginal productivity of aggregate labor:  $r_t = f'(k_t)$  and  $w_t = f(k_t) - k_t f'(k_t)$ , where  $k \equiv K/L$  and  $f(k) \equiv F(k, 1)$  denote, respectively, the capital intensity and the per-worker average productivity. Assumption 2 implies that  $f$  is increasing and concave.

**Monetary authority.** The monetary authority is assumed to follow a constant money growth rule. This means that, in each period  $t$ , the money supply is given by  $M_t^s = \mu^t M_0^s$ , where  $\mu > 0$  is the constant money growth factor, and  $M_0^s$  is the amount of nominal balances available in period zero. Money created at each period is injected to the economy through lump-sum transfers  $\tau_t = (\mu - 1) M_t^s$  to the consumers. We further assume:

**Assumption 3** *the money growth factor is strictly larger than the discount factor:  $\mu > \beta$*

As we will show in the sequel, Assumption 3 ensures that constraint (1) binds in a neighborhood of the steady and that all the relevant variables of the model evaluated at the steady state are strictly positive.

## 2.1 Intertemporal equilibrium

Since the population size is normalized to one, it follows that equilibrium in factors market is obtained by setting  $K_t = k_t$  and  $L_t = l_t$  for every  $t \geq 0$ . When constraint (1) binds, money market equilibrium implies

$$\frac{p_{t+1} c_{t+1} + k_{t+2} - \Delta k_{t+1}}{p_t c_t + k_{t+1} - \Delta k_t} = \frac{M_{t+1}}{M_t} = \mu$$

for every  $t \geq 0$ . Eventually, Walras law ensures good market clearing in each period. By opportunely manipulating the first order conditions (2)-(4) and the equilibrium ones, the intertemporal equilibrium of the economy can be described in terms of the dynamic evolution of the vector  $(k_t, c_t, \lambda_t)$ .

**Definition 1** *An interior intertemporal equilibrium with perfect foresight is a strictly positive sequence  $\{k_t, c_t, \lambda_t\}_{t=0}^{\infty}$  satisfying, for every  $t \geq 0$ , equations*

$$u'(c_t)/\beta - u'(c_{t+1})\Delta = \lambda_{t+1}f'(k_{t+1}) \quad (6)$$

$$q\lambda_t f(k_t)\mu/\beta + (1-q)\lambda_{t+1}f(k_{t+1}) = u'(c_{t+1})f(k_{t+1}) \quad (7)$$

$$c_t + k_{t+1} - \Delta k_t = f(k_t) \quad (8)$$

*subject to the initial money supply  $M_0 > 0$ , the initial endowment of capital  $k_0 > 0$  and the transversality condition (5).*

## 2.2 Steady state analysis

Our first goal is to prove the existence and uniqueness of the deterministic steady state of the dynamic system defined by equations (6)-(8). Actually, we are able to provide the explicit expressions of the steady state values of  $k$ ,  $c$ , and  $\lambda$ , by simply dropping the time index from (6)-(8) and by opportunely rearranging terms. Proceeding in such a way, we get the following expressions:  $k = f'^{-1}(r)$ ,  $c = (r/\alpha - \delta)k$ ,  $\lambda = u'(c)/\tilde{q}$ , where  $r = \rho\tilde{q}$ ,  $\rho \equiv 1/\beta - \Delta > 0$  and  $\tilde{q} \equiv 1 + q(\mu/\beta - 1) > 1$ . In the following  $\alpha \equiv kf'(k)/f(k) \in (0, 1)$  and  $\sigma = -u'(c)/[cu''(c)] > 0$  will denote, respectively, the capital share in total income and the elasticity of intertemporal substitution evaluated at the steady state.

It is easily verifiable that these stationary values are strictly positive under the domain of validity of Assumption 3. For constraint (1) to be locally binding, the Lagrange multiplier  $\nu$ , evaluated at the steady state, must be strictly positive, which means, in the light of the first order condition (3), that  $u'(c) - \lambda$  must be positive. Straightforward computations show that this is actually true if and only if Assumption 3 does hold. It follows that equations (6)-(8) describe intertemporal equilibrium of the economy in a sufficiently small neighborhood of the stationary solution.

It is also worthwhile to emphasize that, despite the fact that the supply of labor is fixed, money is not superneutral in this economy. From the steady state values given above, it is easy to show that a higher money growth rate decreases consumption and capital per capita at the steady state. As emphasized by Stockman (1981) and Abel (1985), who demonstrated the same kind of results within a similar economy with full cash-in-advance constraint on consumption and investment, this contractionary effect of higher money growth is due to the tax imposed by inflation on the cost of capital investment. Of course, compared to these previous papers, this contractionary effect is mitigated when  $q$  is relatively small.

After having provided the expressions (6)-(8) describing the dynamics of the economy, we can now go through the stability analysis.

## 3 Stability analysis

In order to study the occurrence of (local) indeterminacy, we follow the usual procedure consisting in analyzing the stability of the deterministic dynamics around the steady state  $(k, c, \lambda)$ . We linearize system (6)-(8) to obtain  $J_1 v_{t+1} = J_0 v_t$ , where  $J_0$ ,  $J_1$  are Jacobian matrixes evaluated at the steady state and  $v_t \equiv (dk_t/k, dc_t/c, d\lambda_t/\lambda)^T$ . The study of the characteristic polynomial  $P_0(\lambda) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$  of the forward looking Jacobian matrix  $J \equiv J_1^{-1}J_0$  enables us to locate the eigenvalues  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  with respect to the unit circle. More precisely, we study the sign of the polynomial evaluated at  $-1$ ,  $0$ ,  $1$ , to locate its intersections with the axis of abscissas: this allows, when the eigenvalues are real, to characterize their modulus and sign. For sake

of simplicity, instead of studying the characteristic polynomial  $P_0$ , we study the linear transformation  $P_1 = \theta_0 P_0$ , where  $\theta_0 \equiv (\Delta - \varkappa/\beta) \tilde{q}/\sigma < 0$ , with  $\varkappa \equiv (\beta/\mu) (\tilde{q}/q)$ , is a reduced parameter which is always greater than one. Therefore, we obtain  $P_1(0) = -(r + \Delta) \tilde{q}/(\beta\sigma)$ ,  $P_1(1) = r\varkappa(1 - \alpha)(r/\alpha - \delta)$  and  $P_1(-1) = \theta_2/\sigma - \theta_1$ , where  $\theta_1 \equiv r(r/\alpha - \delta)[2 - (1 - \alpha)\varkappa]$  and  $\theta_2 \equiv 2\tilde{q}(1 + \Delta + r)[(\varkappa - 1)/\beta - \Delta]$ .

The next Proposition is the main result of the paper and characterizes the local stability of the system defined by equations (6)-(8). It is shown that all the eigenvalues are real, that there exists one which is always stable and one that lies always outside the unit circle. By contrast, the stability of the third eigenvalue depends upon the structural parameters of the model. Namely, when the amplitude of the financial constraint applying on consumption and investment expenditures is low enough (smaller than a certain threshold  $q_1$ ), the third eigenvalue belongs to  $(-1, 0)$  and thus equilibrium is locally indeterminate for whatever elasticity of intertemporal substitution in consumption. By contrast, when the amplitude of the liquidity constraint is high enough (greater than a second threshold  $q_2 > q_1$ ), the third eigenvalue belongs to  $(-\infty, -1)$  and the equilibrium is locally determinate. In the intermediate case  $q_1 < q < q_2$ , indeterminacy prevails only for strong enough income effects in intertemporal substitution in consumption (or high relative risk aversion), as measured by parameter  $\sigma$ .

To be specific, let  $q_1$ ,  $q_2$  and  $\sigma^*$  be, respectively, the solutions of  $\theta_1 = 0$ ,  $\theta_2 = 0$  and  $P_1(-1) = 0$ , that is  $\sigma^* = \theta_2/\theta_1$ . We obtain

$$q_1 \equiv \frac{1}{1 + \frac{\mu}{\beta} \frac{1+\alpha}{1-\alpha}}$$

$$q_2 \equiv \frac{1}{1 + \mu\Delta} \tag{9}$$

$$\sigma^* = \left( \frac{1 + \Delta + r}{\rho(r/\alpha - \delta)} \right) \left( \frac{(\varkappa - 1)/\beta - \Delta}{1 - \varkappa(1 - \alpha)/2} \right) \tag{10}$$

It is straightforward to verify that  $0 < q_1 < q_2 < 1$ . Using these definitions, we can now formulate:

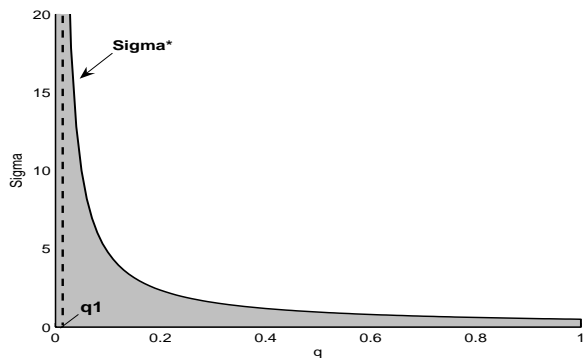
**Proposition 1** *Consider the dynamic system under perfect foresight defined by (6)-(8) and its corresponding (unique) steady-state. The three eigenvalues associated to this system are real, with  $\lambda_1 < 0 < \lambda_2 < 1 < \lambda_3$ . Moreover:*

- (1) *If  $0 < q < q_1$ , then  $\lambda_1 > -1$  and the steady-state is locally indeterminate.*
- (2) *If  $q_1 < q < q_2$ : For  $\sigma < \sigma^*$ ,  $\lambda_1 > -1$  and the steady state is locally indeterminate; For  $\sigma > \sigma^*$ ,  $\lambda_1 < -1$  and the steady state is locally determinate. In addition, when  $\sigma$  goes through  $\sigma^*$ , the steady state undergoes a flip bifurcation.*
- (3) *If  $q_2 < q \leq 1$ , then  $\lambda_1 < -1$  and the equilibrium is locally determinate.*

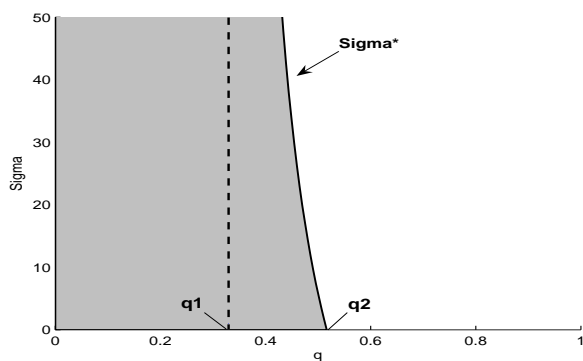
**Proof.** See the Appendix. ■

Proposition 1 generalizes the case studied by Stockman (1981) and Abel (1985) of an economy with full liquidity constraint on aggregate income. The

main results emphasized by this proposition are illustrated in the lower panel of Figure 1, where the regions delimited by the critical parameters  $q_1$ ,  $q_2$  and  $\sigma^*$  are depicted in the  $(q, \sigma)$  plane. For comparison purposes, the upper panel of this figure also displays these results for the alternative reference economy where the liquidity constraint applies to consumption only.<sup>3</sup>



Liquidity constraint on consumption only



Liquidity constraint on consumption and investment

Figure 1: Indeterminacy regions in the two benchmark economies

One striking feature emerging from Figure 1 is that the inclusion of investment in the liquidity constraint has dramatic effects on the stability properties of the model, as it modifies considerably the range of parameters values giving rise to indeterminacy. However, these effects do not point uniformly toward an increased (or a decreased) instability when the liquidity constraint applies to a fraction of total income. As may be seen, the critical threshold  $q_1$  below which the steady-state is always indeterminate, which was small and even close to zero in the consumption-based constraint, is strongly increased by the inclusion of

<sup>3</sup>We have used in both graphics the same standard annual calibration for the structural parameters:  $\beta = 0.95$ ,  $\mu = 1.04$ ,  $\delta = 0.1$  and  $\alpha = 0.3$ . See Bosi and Magris (2003) for an explicit analytical derivation of the critical values in the consumption-based liquidity constraint case.

investment in that constraint (reaching a value of  $q_1 = 0.34$ ). But, at the same time, this inclusion of investment also creates a *second threshold*  $q_2$  above which the steady-state is *always determinate* (threshold which, using the standard annual calibration mentioned above, is approximately around 0.51). Finally, when the share  $q$  lies in the  $[q_1, q_2]$  interval, indeterminacy depends on the relative risk aversion of consumers. The critical value for  $\sigma$  below which the steady-state is indeterminate is defined implicitly by a locus  $\sigma^* = \sigma(q)$  which tends to decrease from  $+\infty$  to 0 as  $q$  increases from  $q_1$  to  $q_2$ . Note however that, for the standard calibration considered, this locus has a slope which is very steep in the  $(q, \sigma)$  plane. This implies that, in most of this interval, indeterminacy prevails for a large range of values of  $\sigma$  including all empirically plausible estimates.

It results from this analysis that the fraction of consumption and investment purchases requiring cash is the crucial parameter to determine whether constant money growth rules are likely to prone stability and rule out sunspot fluctuations in this economy. From an empirical perspective, it is interesting to observe that this parameter can be given another interpretation in terms of money velocity. Indeed, equation (1) implies, when it is binding, that  $M_t/p_t = q(c_t + k_{t+1} - \Delta k_t) \equiv qy_t$ , where  $y_t$  is aggregate output. Hence, the parameter  $q$  appears as the (inverse of) the steady-state velocity of money, a parameter about which the literature provides empirical estimates. Interpreting money to be the money base, Carlstrom and Fuerst (2003) report a value of  $q = 1/3$ , which falls in the indeterminacy region depicted in Figure 1. However, Orphanides and Porter (2000) provide a more detailed empirical analysis using data on M2 which leads to slightly different estimates. One central result in their paper, which is particularly interesting from our perspective, is that they find that the steady state velocity of money has tended to *increase* during the recent decades, rising from a value of 1.70 in the 60's and 70's to a value higher than 2 in the 90's. This corresponds, respectively, to estimates for  $q$  of 0.59 and 0.5, suggesting that the economy was in the determinacy region during most of the recent decades. Note however that the second value falls very close to  $q_2$ , the second threshold below which the equilibrium quickly becomes indeterminate. Hence, if the upward trend for money velocity were to be confirmed in the future, the model suggests that adopting a money growth targeting rule might not easily achieve stability – a topic which is already a concern for the currently used interest rate policy rule.

## 4 Conclusion

In this paper, we have studied the stability properties of a one-sector productive economy under constant money growth rules, when agents are subject to a partial cash-in-advance constraint applying on consumption and investment purchases. It appears from our analysis that taking liquidity constraints on investment into account have considerable effects on the range of parameters values giving rise to local indeterminacy. In particular, we found that money growth targeting rules uniformly ensure equilibrium determinacy when the liq-

liquidity constraint is strong, but may favor the emergence of expectation-driven fluctuations if that constraint is weak. In the data, the partial liquidity constraint may be approximated by the inverse of the steady-state velocity of money. While the postwar US data seem to suggest that the economy was until recently in the determinacy region, the recurrent tendency for money velocity to increase in recent decades, traducing the increase in transaction facilities due to the introduction of more liquid assets and new forms of payments, may well reverse soon this conclusion.

## 5 Appendix

**Proof of Proposition 1.** First, recall that the characteristic polynomial  $P_1(\lambda)$  is a continuous function and that its domain is connected. We observe that  $P_1(0) < 0$  and  $P_1(1) > 0$ . Moreover,  $\theta_0 < 0$  implies  $\lim_{\lambda \rightarrow +\infty} P_1(\lambda) = -\infty$ . Then  $\lambda_1 < 0 < \lambda_2 < 1 < \lambda_3$ . In order to know whether  $\lambda_1$  is greater than  $-1$ , we solve the inequality  $P_1(-1) > 0$ . If  $\theta_1 < 0$ , then  $\theta_2 > 0$  and  $\theta_2/\sigma - \theta_1 = P_1(-1) > 0$ . If, on the other hand,  $\theta_1 > 0$ , then  $P_1(-1) > 0$ , if and only if  $\sigma < \sigma^* \equiv \theta_2/\theta_1$ . In other words,  $\lambda_1 > -1$ , if and only if either  $\theta_1 < 0$  or  $\sigma < \sigma^*$ . But  $\theta_1 < 0$  is equivalent to  $q < q_1$ , while  $\sigma^* < 0$ , with  $\theta_1 > 0$ , to  $\theta_2 < 0$ , that is  $q > q_2$ . It follows that for  $q < q_1$ ,  $\lambda_1 > -1$  whatever  $\sigma$ , while  $q_1 < q < q_2$  implies  $\lambda_1 > -1$ , if and only if  $\sigma < \sigma^*$ . Eventually, if  $q > q_2$ ,  $\lambda_1 < -1$ . A flip bifurcation arises when  $\sigma = \sigma^*$ . ■

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